1.0

4.5

2.8   2.5

3.2

2.2

3.6

4.0   2.0

1.1

1.8

1.25   1.4   1.6

MICROCOPY RESOLUTION TEST CHART

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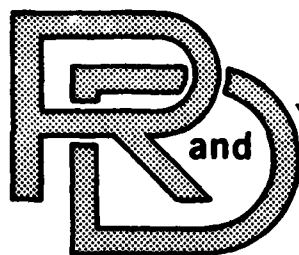
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TECHNICAL REPORT

NO. 12512

A METHODOLOGY FOR
ESTIMATING MISSION
RELIABILITY

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by L. R. Lamberson & K. C. Kapur

**U.S. ARMY TANK-AUTOMOTIVE
RESEARCH AND DEVELOPMENT COMMAND
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This research is specifically directed towards the application of techniques for estimating mission reliability where data is generated from proving ground or actual field testing. Three methodologies are refined for the typical Army vehicle testing situation. These are (i) failure data analysis, (ii) use of the exponential distribution, and (iii) use of the binomial distribution. Applicable approximations are explored for easy application along with typical examples of their use.

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A METHODOLOGY FOR ESTIMATING
MISSION RELIABILITY

by

Kailash C. Kapur
and
Leonard R. Lamberson
Wayne State University
Detroit, Michigan 48202

Table of Contents

A METHODOLOGY FOR ESTIMATING MISSION RELIABILITY

0.0 Introduction

This research is specifically directed towards estimating vehicle mission reliability where data is generated from proving ground or actual field testing. In proving ground testing the mission profile is assumed to be an accurate representation of field usage. Thus, the mission reliability estimate will be indicative of that which should be achieved by the vehicle in actual use conditions. The typical vehicle testing situation which is the basis for estimation will now be delineated.

In proving ground or field testing there is normally a target test bogy which is a specified mileage (or kilometers) for the vehicle. As the vehicle traverses the proving ground, failures are generated*. This situation is shown pictorially in Figure 1, below. The vehicle will sometimes not exactly meet the test bogy

Figure 1   Proving Ground Vehicle Testing Situation

*Actually, failure incidents are generated which are subsequently scored by an offical Scoring Committee.

due to the randomness associated with a product testing program.

This testing situation is analogous to a random variate generator where each variate (t_{ij}) is generated by the vehicle after a repair. It will be assumed that the distribution regenerates after each failure. A further, and more complex, approach would be to assume that the distribution does not regenerate.

Specific approaches to mission reliability estimation will be adapted to this testing situation. The emphasis was to develop and present easy-to-use techniques for mission reliability estimation. The approaches considered in this report are: (i) failure data analysis, (ii) exponential distribution, and (iii) binomial distribution. Application of each technique is covered in the following sections.

SECTION I

FAILURE DATA ANALYSIS BASED ON

TOTAL TIME ON TEST PLOTS

Failure Data Analysis Based on

Total Time on Test Plots

1.0 Introduction

In order to compute mission reliability, we present the
adaptation of a new method in this section. The methods pre-
sented here are non-parametric in nature in the sense that
they do not assume the nature of the underlying failure dis-
tribution. These methods are based on the concept of the
total time on test transform. The mathematical concept of
the total time on test transform was introduced in Chapters
5 and 6 of reference [3]*. The total time on test plot is
useful in choosing a probabilistic model in terms of a failure
distribution to represent the failure behavior of the data.
Thus, by using the methodology presented here, the reliability
engineer can make a decision regarding which failure distribu-
tion "best" represents the data. The approach is graphical in
nature and thus is very easy to implement in the Army's environ-
ment. The procedures presented will help the reliability en-
gineer to plot the test data in a systematic way and develop
suitable failure models. Once a decision is made on the selec-
tion of the failure model, mission reliability can be easily
computed.

It is not always necessary to develop the failure model.
From the plots presented here, a hazard function can be easily
developed and hence using the knowledge about the hazard

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*See page 71.

function we can easily compute the mission reliability. The graphical estimation of the failure rate or hazard rate function is used to compute the mission reliability. The methods discussed here can analyze incomplete data. The total time on test plot has a theoretical basis, and the mathematical theory for these plots has been developed [8]. These plots converge to a transform of the underlying probability distribution as the sample size increases.

2.0 Concept of the Total Time on Test Transform

Mission reliability of a military system is evaluated by testing the system and observing the times when failures occurred. The underlying distribution is generally not known. Hence, nonparametric methods will be of tremendous value to evaluate the mission reliability.

In order to develop the concept of the total time on test transform, we first present the method of evaluating reliability based on the assumption that the underlying failure distribution is exponential [See also page 47].

Suppose we have an ordered sample from an exponential distribution

$$f(t) = \lambda e^{-\lambda t} \quad , \quad t \geq 0, \lambda > 0 \tag{1}$$

$$\text{or} \quad f(t) = \frac{1}{\theta} e^{-t/\theta} \quad , \quad t \geq 0, \theta > 0 \tag{2}$$

where $\lambda = \frac{1}{\theta}$ = failure rate

and θ = mean life or MTBF

We write the ordered sample as

$$0 = t_{n0} \leq t_{n1} \leq \cdots \leq t_{nn} \tag{3}$$

where the index n means that n items are placed on test and in the above case, we observe all the n failures. In case, we only observe r failures when we have n items on test, then the ordered sample will be written as

$$0 = t_{n0} \leq t_{n1} \leq \cdots \leq t_{nr} \tag{4}$$

It is clear that the total time on test is the sum of all observed complete and incomplete life times and is given by

$$T(t_{nr}) = \text{total time on test (a function of } t_{nr})$$
$$= n\, t_{n1} + (n-1)(t_{n2} - t_{n1}) + \ldots + (n-r+1)(t_{nr} - t_{n,r-1})$$

6

$$= \sum_{i=1}^{r} t_{ni} + (n-r)t_{nr} \tag{5}$$

Based on total time on test, we can estimate θ, the MTBF for the system. It is well known [10,11,12] that

$$\hat{\theta}_{r,n} = \frac{1}{r} T(t_{nr}) \tag{6}$$

is the maximum likelihood estimator of θ and is the unique minimum variance and unbiased estimator of θ. The above estimator has been extensively used for the evaluation of system reliability and the development of confidence bounds with the associated mathematics given in [14]. However, it must be remembered that this estimator is based on the assumption that the underlying failure distribution is exponential.

Now, we present the development of the concept of total time on test where the underlying distribution is not known and is any arbitrary distribution. Given the ordered sample information (Eq. (3)), the empirical distribution for the underlying arbitrary distribution may be written as

$$F_n(t) = \begin{cases} 0 & , \quad t \le t_{n1} \\ \frac{i}{n} & , \quad t_{ni} \le t \le t_{n,i+1} \\ 1 & , \quad t \ge t_{nn} \end{cases} \tag{7}$$

The use of the above empirical distribution is well known to the engineer. Now, we develop the concept of total time on test based on the above empirical distribution (See Figure 1). Let us define

$$F_n^{-1}(u) = \inf\{t | F_n(t) > u\} \tag{8}$$

7
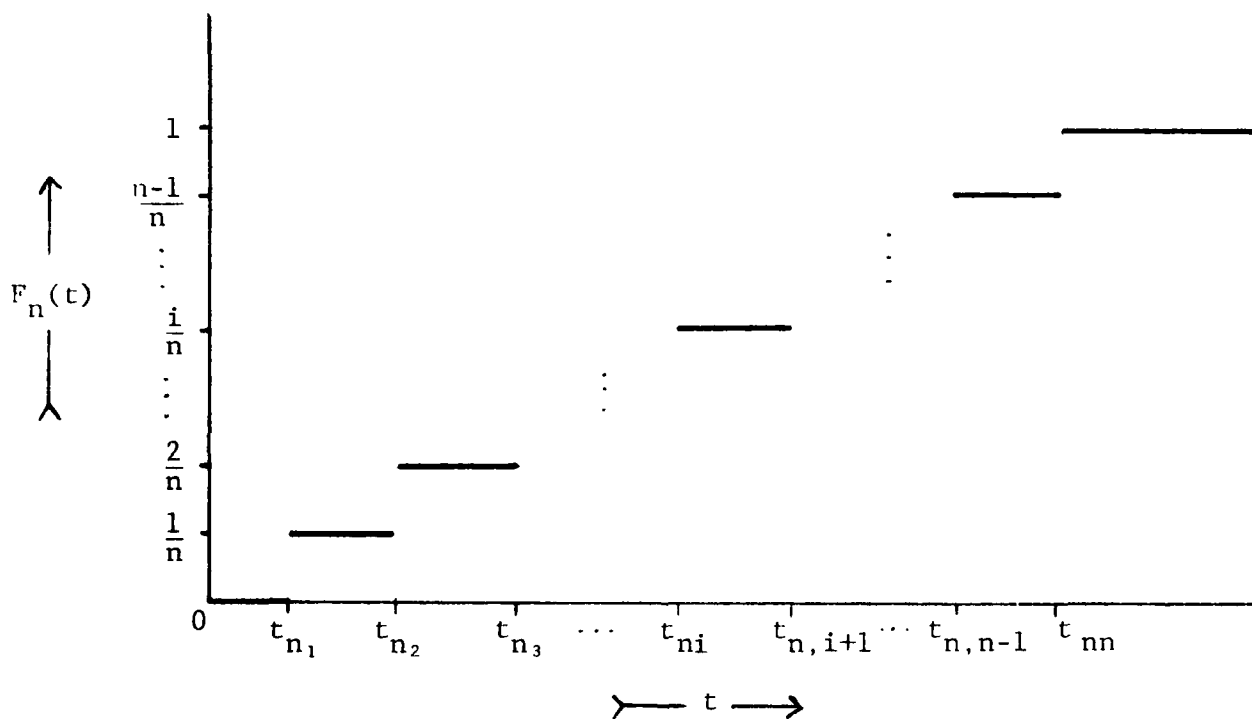
Figure 1 - <u>Empirical Distribution Based on Test Data</u>

8

For example, if $\frac{1}{n} \le u < \frac{2}{n}$, then $F_n^{-1}(u) = t_{n2}$, and in general for $\frac{i}{n} \le u < \frac{i+1}{n}$, we have

$$F_n^{-1}(u) = t_{n,i+1}, \quad i = 0,1,2,\ldots,n-1. \tag{9}$$

Now, let us compute the following integral

$$\int_0^{F_n^{-1}(\frac{r}{n})} (1 - F_n(u)) \, du = \int_0^{F_n^{-1}(\frac{r}{n})} \bar{F}_n(u) \, du \tag{10}$$

where $\bar{F}_n(u) = 1 - F_n(u)$. Using eqs. (7) and (9), we have

$$\int_0^{F_n^{-1}(\frac{r}{n})} \bar{F}_n(u) \, du = \sum_{j=1}^{r} [1 - \frac{j-1}{n}] [t_{nj} - t_{n,\,j-1}]$$

$$= \frac{1}{n} \sum_{j=1}^{r} (n - j + 1)(t_{nj} - t_{n,j-1})$$

$$= \frac{1}{n} [n \, t_{n1} + (n-1)(t_{n2}-t_{n1}) + \ldots + (n-r+1)$$

$$(t_{nr}-t_{n,r-1})]$$

$$\text{or} \quad \int_0^{F_n^{-1}(\frac{r}{n})} \bar{F}_n(u) \, du = \frac{1}{n} T(t_{nr}) \tag{11}$$

The last equaiton follows from eq. (5). Let us now consider the limiting case when $n \to \infty$ and $\frac{r}{n} \to v$. In this case, it is well known that

$$\lim_{\substack{n \to \infty \\ \frac{r}{n} \to v}} \int_0^{F_n^{-1}(\frac{r}{n})} \bar{F}_n(u)\,du = \int_0^{F^{-1}(v)} \bar{F}(u)\,du \qquad (12)$$

uniformly in $v, (0 \le v \le 1)$. Here $F(\cdot)$ is the underlying failure distribution function and the above statement means that as the sample size n approaches infinity, the empirical distribution given by (7) converges uniformly to the actual distribution $F(\cdot)$.

Let us define

$$H_F^{-1}(v) \triangleq \int_0^{F^{-1}(v)} \bar{F}(u)\,du \qquad , \qquad 0 \le v \le 1 \qquad (13)$$

as the total time on test tranform.

For any distribution function $F(t)$, it is well known that

$$H_F^{-1}(1) = \int_0^{F^{-1}(1)} \bar{F}(u)\,du = \text{ expected value of r.v.t.} \qquad (14)$$

Thus, the basic relationship between the total time on test transform given by (13) and total time on test $T(t_{nr})$ given by (5) is

$$\lim_{\substack{n \to \infty \\ \frac{r}{n} \to v}} \frac{1}{n}\, T(t_{nr}) = \int_0^{F^{-1}(v)} \bar{F}(u)\,du = H_F^{-1}(v) \qquad (15)$$

Let us consider the example of an exponential distribution. We have

$$G(t) = 1 - e^{-t/\theta} , \qquad t \ge 0, \qquad \theta > 0 \qquad (16)$$

10

Hence, the total time on test transform is given by

$$H_G^{-1}(v) = \int_0^{G^{-1}(v)} \bar{G}(u) \, du$$

$$= \int_0^{G^{-1}(v)} e^{-u/\theta} \, du$$

$$= \int_0^{G^{-1}(v)} \theta \, dG(u)$$

$$= \theta \int_0^{G^{-1}(v)} dG(u)$$

$$= \theta v, \qquad 0 \le v \le 1 \tag{17}$$

And, we have

$$\frac{H_G^{-1}(v)}{H_G^{-1}(v)} = \frac{\theta v}{\theta} = v, \qquad 0 \le v \le 1 \tag{18}$$

Thus, by scaling the total time on test transform by its mean, the scaled total time on test transform is transformed into a 45° line on the interval [0,1].

11

2.1 Relationship of Total Time on Test Transform to Failure Rate

The total time on test transform is related to the failure rate for the underlying failure distribution. Let us remember that the failure rate or hazard rate $h(t)$ is given by [see 14]

$$h(t) = \frac{f(t)}{R(t)} \tag{19}$$

$$= \frac{f(t)}{1-R(t)} = \frac{f(t)}{\bar{F}(t)} \tag{20}$$

and

$$R(t) = \exp\left[-\int_{o}^{t} h(u)\, du\right] \tag{21}$$

Thus, we see that $f(t)$, $F(t)$, $R(t)$ and $h(t)$ are all related to each other and the knowledge about any one of them determines the remaining three. Now, we observe that

$$\frac{d}{dv} H_F^{-1}(v)\bigg|_{v = F(t)} = \frac{d}{dv} \int_{o}^{F^{-1}(v)} \bar{F}(u)\, du\bigg|_{v = F(t)}$$

$$= \frac{1-v}{f[F^{-1}(v)]}\bigg|_{v = F(t)}$$

$$= \frac{1 - F(t)}{f[F^{-1}(F(t))]} = \frac{1-F(t)}{f(t)}$$

$$= \frac{R(t)}{f(t)} = \frac{1}{h(t)} \tag{22}$$

Thus, the rate of change of the total time on test transform is equal to one divided by the associated failure rate. This means that if we have a plot of total time on test transform, we can estimate the failure rate from it.

If the underlying failure distribution F has an increasing failure rate (IFR), i.e. $h(t)$ increases with t, then the rate of change of $H_F^{-1}(v)$ is decreasing and hence $H_F^{-1}(v)$ is a concave function in v, $(0 \leq v \leq 1)$. On the other hand, if the underlying distribution has a decreasing failure rate (DFR), i.e. $h(t)$ decreases with t, then $H_F^{-1}(v)$ is a convex function in v, $(0 \leq v \leq 1)$. This property about the nature of the total time on test transform can be used to understand the behavior of the failure rate for the underlying system. This will give us an idea whether the failure rate is increasing or decreasing during a certain time period.

From eq. (22), it is also clear that H_F determines F if F is absolutely continuous. Since absolutely continuous distributions are dense in the class of all failure distributions, we see that $H_F^{-1}(v)$ determines F in general. This is an important property because it tells us that $H_F^{-1}(v)$ determines F and we can use the total time on test transform for identification of the failure distribution model for the system.

One of the important concepts in reliability theory is the concepts of failure rate on an average. A failure distribution F is said to have increasing failure rate average (IFRA) if the cumulative failure rate on an average with respect to time increase with time. Thus, the distribution F is IFRA if

13

$$\frac{1}{t} \int_0^t h(u) \, du \uparrow t \geq 0 \tag{23}$$

From eq. (21), it is clear that

$$\int_0^t h(u) \, du = - \ln R(t) = - \ln \bar{F}(t) \tag{24}$$

Hence, we can restate the definition that distribution F has increasing failure rate average (F is IFRA) if $\dfrac{- \log \bar{F}(t)}{t}$ is increasing (non-decreasing) in $t \geq 0$. Similarly, F has decreasing failure rate average (F is DFRA) if $\dfrac{- \log \bar{F}(t)}{t}$ is decreasing (non-increasing) in $t \geq 0$.

Now we give the behavior of the total time on test transform for IFRA and DFRA distributions. If failure distribution F is IFRA (DFRA), than $H_F^{-1}(v)/v$ is non-increasing (non-decreasing) in v, $0 \leq v \leq 1$ and $H_F^{-1}(v)/H_F^{-1}(v)$ lies above (below) the 45° line. The proof for this statement is as follows.

Let F be IFRA with a probability density function f. Then as given by eq. (23), we have that

$$\frac{1}{t} \int_0^t h(u) \, du \uparrow t \geq 0$$

where $\quad h(t) = \dfrac{f(t)}{\bar{F}(t)}$

Let $S(t) = \displaystyle\int_0^t \bar{F}(u) \, du$, then $S(t)/t$ is always non-increasing in $t \geq 0$ because $\bar{F}(u)$ is always non-increasing in $t \geq 0$. Also

$$\frac{S(t)}{\bar{F}(t)} = \frac{\int_0^t \bar{F}(u) \, du}{\bar{F}(t)} \tag{25}$$

in non-increasing in $t \geq 0$. Let $v = F(t)$ in eq. (25), then then we have that $H_F^{-1}(v)/v$ is non-increasing in $v \geq 0$. Similarly we can prove the result when F is DFRA.

14

3.0 Failure Distribution Model Identification Based on Total Time on Test Transform

It was pointed out in the last section that the total time on test transform $H_F^{-1}(v)$ determines the underlying failure distribution F. We also showed that for the exponential distribution, if we plot the scaled total time on test transform (eq. (18)), we get a 45° line on the interval [0, 1]. Thus, a 45° line indicates that the underlying distribution is exponential. Similarly, we can develop relationships for other distributions.

For the Weibull distribution, we have

$$F(t) = 1 - \exp[-(\tfrac{t}{\theta})^\beta] \quad , \qquad t \geq 0 \tag{26}$$

where $\beta > 0$ is the shape parameter and $\theta > 0$ is the scale parameter or the characteristic life. We also showed that (see eq. (14))

$$H_F^{-1}(1) = \text{expected life for Weibull r.v.}$$
$$= \theta \ \Gamma(1 + \tfrac{1}{\beta}) \tag{27}$$

The results for the Weibull distribution are given in references [14,17,18,24, and 25]. In this case,

$$H_F^{-1}(v) = \int_0^{F^{-1}(v)} \exp[- (\tfrac{t}{\theta})^\beta]dt \tag{28}$$

Based on eqs. (28) and (27), we can plot the scaled total time on test transforms for Weibull distribution. The approximate shapes for these plots for different values of the shape parameter β are given in Figure 2. It is clear that the plots
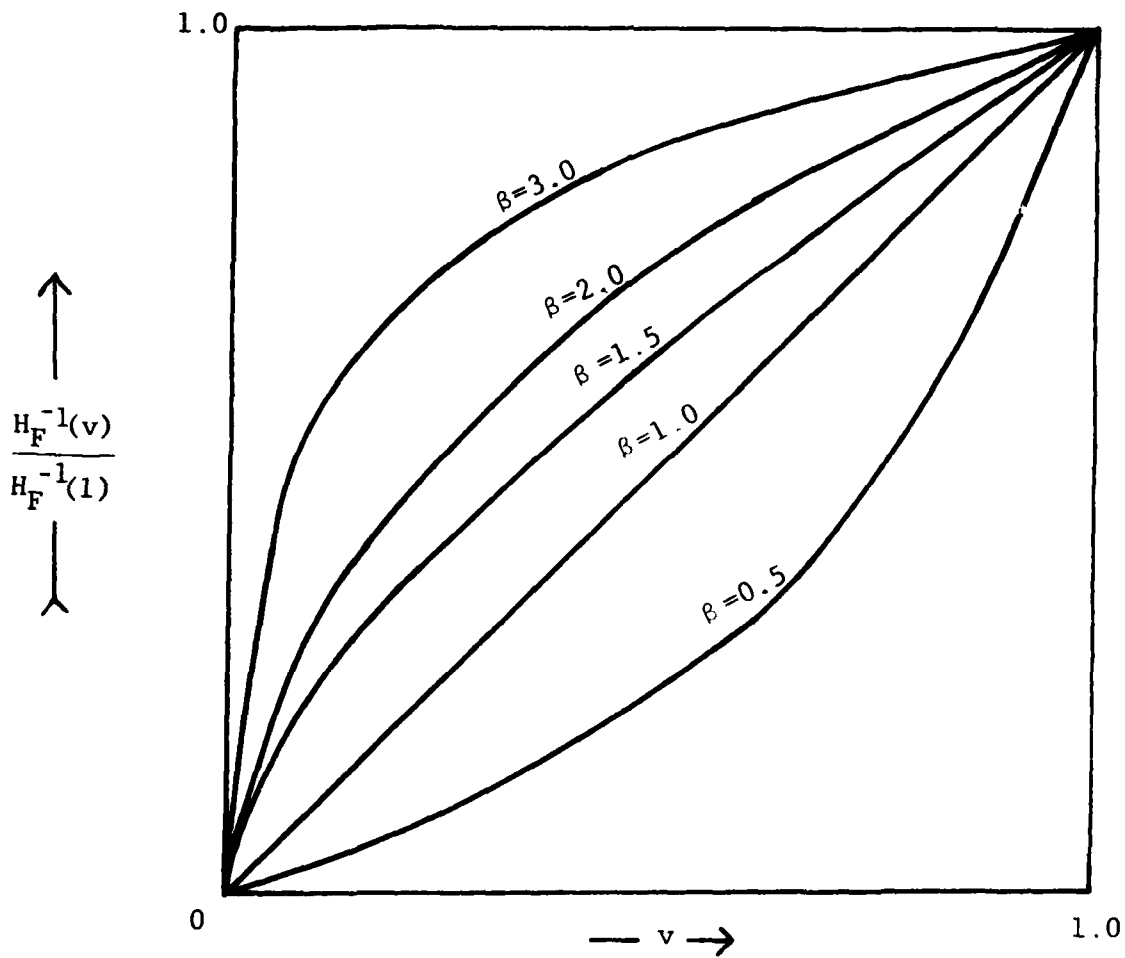
Figure 2 - <u>Scaled Total Time on Test Transform for</u>
<u>Weibull Distribution</u> (β is the shape parameter)

are convex which means that we have decreasing failure rate for $\beta < 1$ and concave which means that we have an increasing failure rate for $\beta > 1$. The exponential distribution is the same as $\beta = 1$ and hence we have a 45° line as proven before.

For the gamma distribution, the probability density function is

$$f(t) = \frac{\lambda^n}{\Gamma(n)} \, t^{n-1} \, e^{-\lambda t} \quad , \qquad t \geq 0, \; n > 0, \; \lambda > 0 \qquad (29)$$

where n is the shape parameter and λ is the scale parameter. Also,

$$H_F^{-1}(1) = \text{expected life for gamma r.v.}$$
$$= \frac{n}{\lambda} \qquad (30)$$

Again, for the gamma distribution, we can plot the scaled total time on test plots as given in Figure 3. These plots are concave functions indicating an increasing failure rate for $n > 1$.

For the lognormal distribution, we have

$$F(t) = \Phi\left[\frac{\ln t - \mu}{\sigma}\right] \qquad (31)$$

where $\Phi [\cdot]$ is the cumulative distribution for the standard normal variable. Again, for the lognormal distribution, μ is the scale parameter and σ is the shape parameter. For each $\sigma > 0$, the scaled total time on test plot crosses the 45° line at most once from above and the approximate shapes for these graphs is given in Figure 4. The abscissa values for these cross-over point decreases with increasing σ values. Thus, for some values of σ, the failure rate for the lognormal distribution is neither always an increasing failure rate nor a decreasing failure rate.
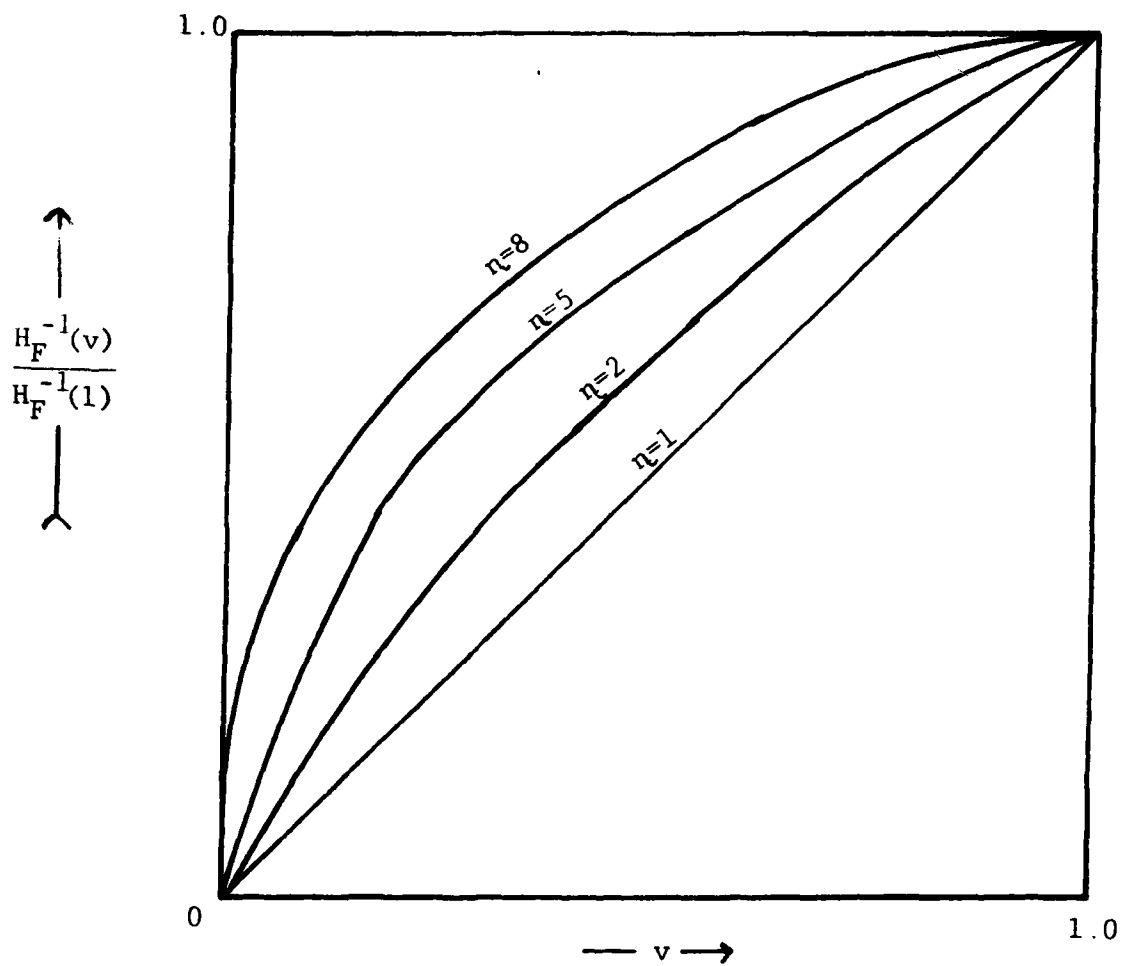
Figure 3 - <u>Scaled Total Time on Test Transform for</u>
<u>Gamma Distribution</u> (η is the shape parameter)

18
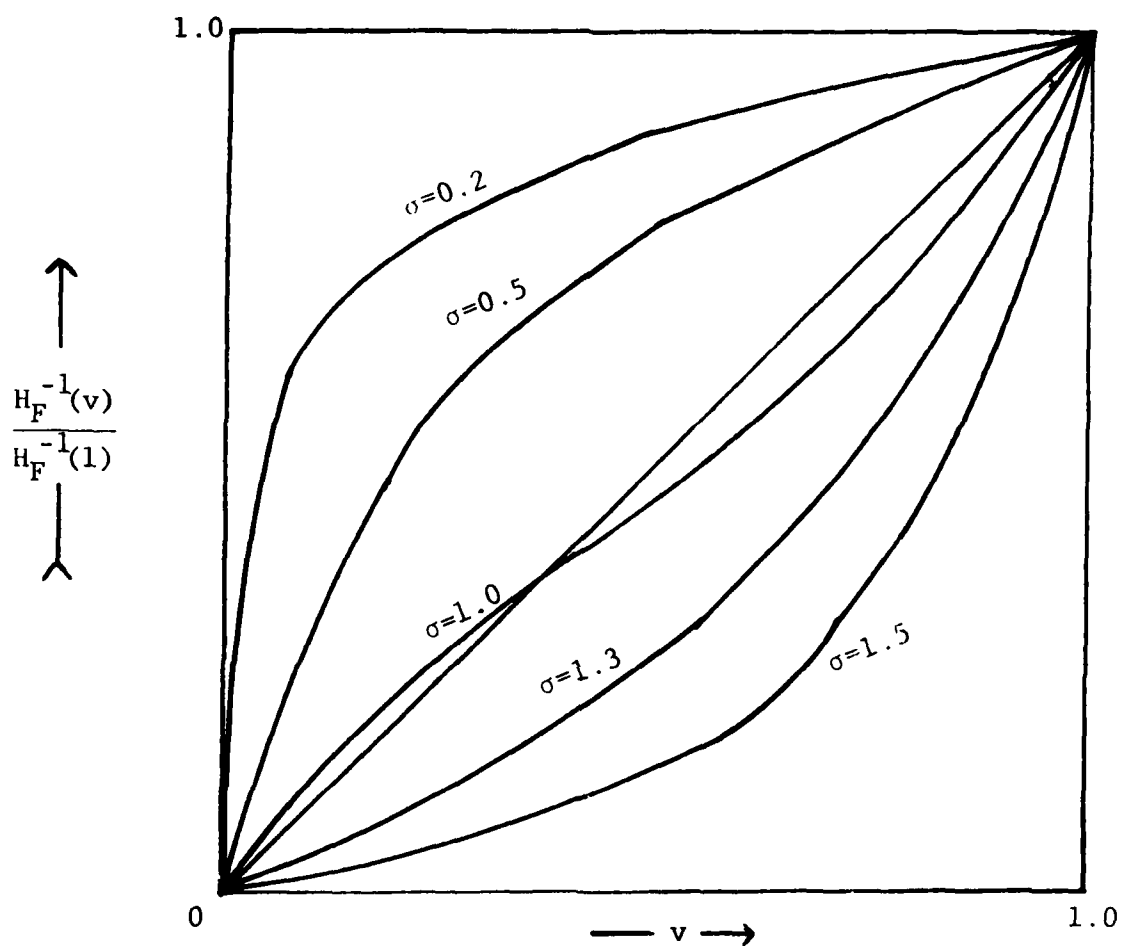
Figure 4 - Scaled Total Time on Test Transform for Lognormal Distribution (σ is the shape parameter)

The approximate shapes of the total time on test plots for the trucated normal distribution are given in Figure 6. For the normal distribution, we always have an increasing failure rate.
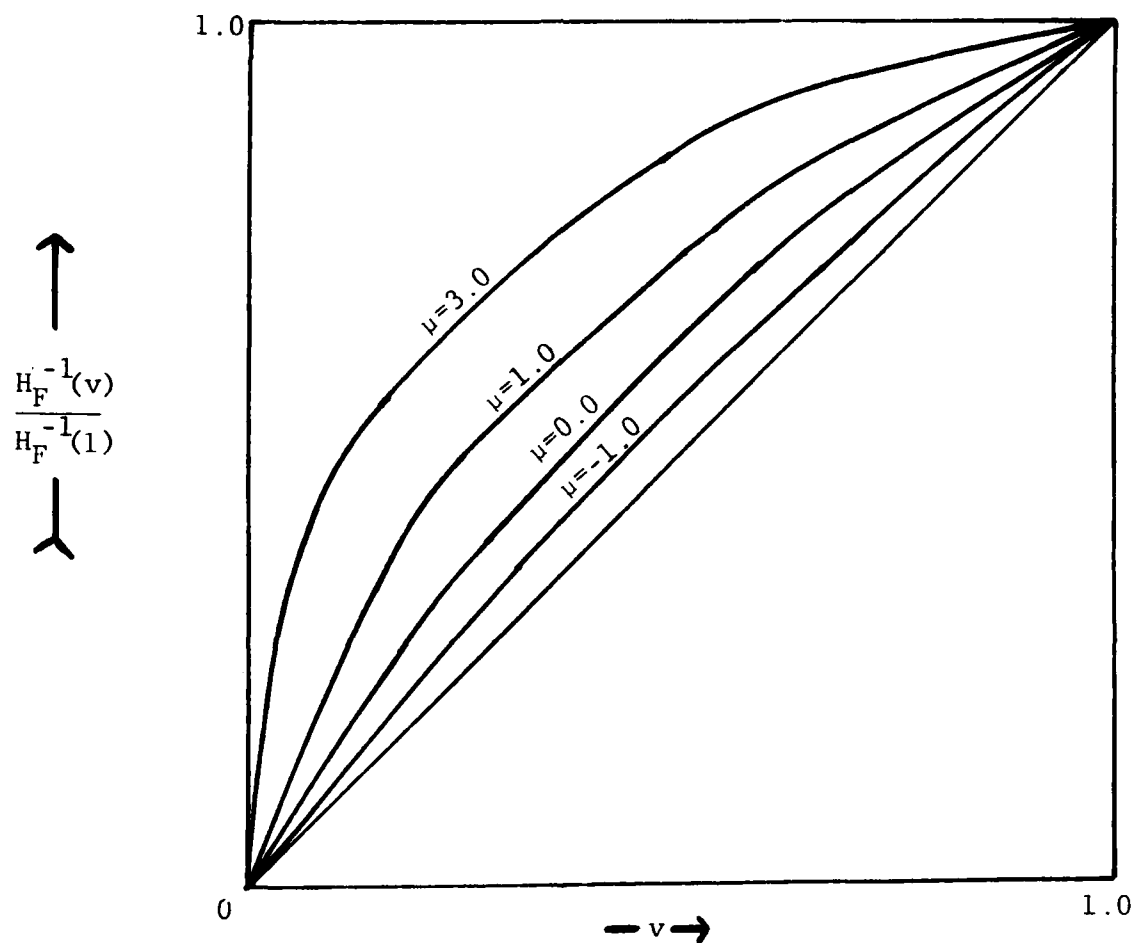
Figure 5 - Scaled Total Time on Test Transform for
Truncated Normal Distribution

4.0 Total Time on Test Plots Based on Test Data

In this section we are going to explain how to plot the total time on test transform when we have test data for the system under study. The plots are similar to the scaled total time on test transforms discussed before. These plots can be used to identify the underlying failure distribution based on the shape of the plots as discussed in the last section. The plots may also be used to evaluate the failure rate and hence the mission reliability by using the relation

$$R(t) = \exp[-\int_{o}^{t} h(u) \, du]. \tag{32}$$

Model identification can be aided by the use of transparent overlays of the scaled total time on test transforms for well known distributions such as the Weibull, lognormal, normal, gamma, etc. We are going to discuss the plots for different test data situations.

4.1 Total Time on Test Plots when all the Failures are Observed

If we put n items on test and observe all the failures, then we have test data as follows:

$$0 = t_{n0} \leq t_{n1} \leq \cdots \leq t_{nn} \tag{33}$$

Thus all the observations are on the age of failure of items all of which are put on life test at time 0. Let $n(u)$ be the number of items working at time $0 \leq u \leq t_{nn}$. Then it is clear that

$T(t)$ = total time on test at time t

$$= \int_0^t n(u) \, du \tag{34}$$

Actually eq. (34) is a finite sum since $n(u)$ is a step function which is nonincreasing.

Also, remember that (from eqs. (13) & (15))

$$H_F^{-1}(v) \triangleq \int_0^{F^{-1}(v)} \overline{F}(u) \, du \approx \frac{1}{n} T(t_{nr}) \tag{35}$$

under the conditions mentioned before, i.e. $n \to \infty$ and $\frac{r}{n} \to v$. Thus, we plot

$$\frac{T(t_{ni})}{T(t_{nn})} \tag{36}$$

versus $\frac{i}{n}$, $i = 1, 2, \ldots, n$. This plot is a function on $[0, \frac{1}{n}, \frac{2}{n}, \ldots, 1]$ to $[0, 1]$ and hence no additional scaling is necessary. This scaled total time on test plot tends to look like the plot of scaled total time on test transform of the underlying distribution, i.e.

$$\frac{H_F^{-1}(v)}{H_F^{-1}(1)} \quad \text{for} \quad 0 \leq v \leq 1 \tag{37}$$

as $n \to \infty$ and $\frac{i}{n} \to v$.

4.1.1 Example (Based on example 10.1 in reference [14])

The data in Table 1 represents a 245 hour vibration simulation test on a half-ton truck. The truck was shaken on a simulator for a total of 245 hours. The time into testing at which a failure occured was recorded.

Table 1 - Vibration Simulation: Failure Time
on a Half-Ton Truck

21.2	74.7	108.6	157.4
47.9	76.8	112.9	164.7
59.2	84.8	127.0	196.8
62.0	91.0	143.9	214.4
74.6	93.3	151.6	218.9

In this case we have $n = 20$ and values in Table 1 are the times t_{ni}, $i = 1, 2, \ldots, 20$. Also $t_{nn} = 218.9$. Hence, we plot $T(t_{ni})/T(t_{nn})$ versus $\frac{i}{n}$ for $i = 1, 2, \ldots, 20$. The values for the plot are given in Table 2. The data given in Table 2 is plotted in Figure 6. The statistical test given in reference [14] based on which the above example is constructed concludes that we cannot contradict the hypothesis that the exponential distribution can be used to model the time to failure for this system.

Table 2 - <u>Values for Scaled Total Time on Test Plot</u>

i	$T(t_{ni})/T(t_{nn})$	i	$T(t_{ni})/T(t_{nn})$
1	0.097	11	0.496
2	0.219	12	0.516
3	0.270	13	0.580
4	0.283	14	0.657
5	0.341	15	0.693
6	0.341	16	0.719
7	0.351	17	0.752
8	0.387	18	0.899
9	0.416	19	0.979
10	0.426	20	1.000

The scaled total time on test transform for the exponential distribution is a straight line and hence we would expect the scaled total time on test plot for our system to lie close to the 45° line. From Figure 6, it is clear that this indeed is the case and the plot croses the 45° line two times. Thus the technique of total time on test plot is consistent with statistically validating the exponential failure model.
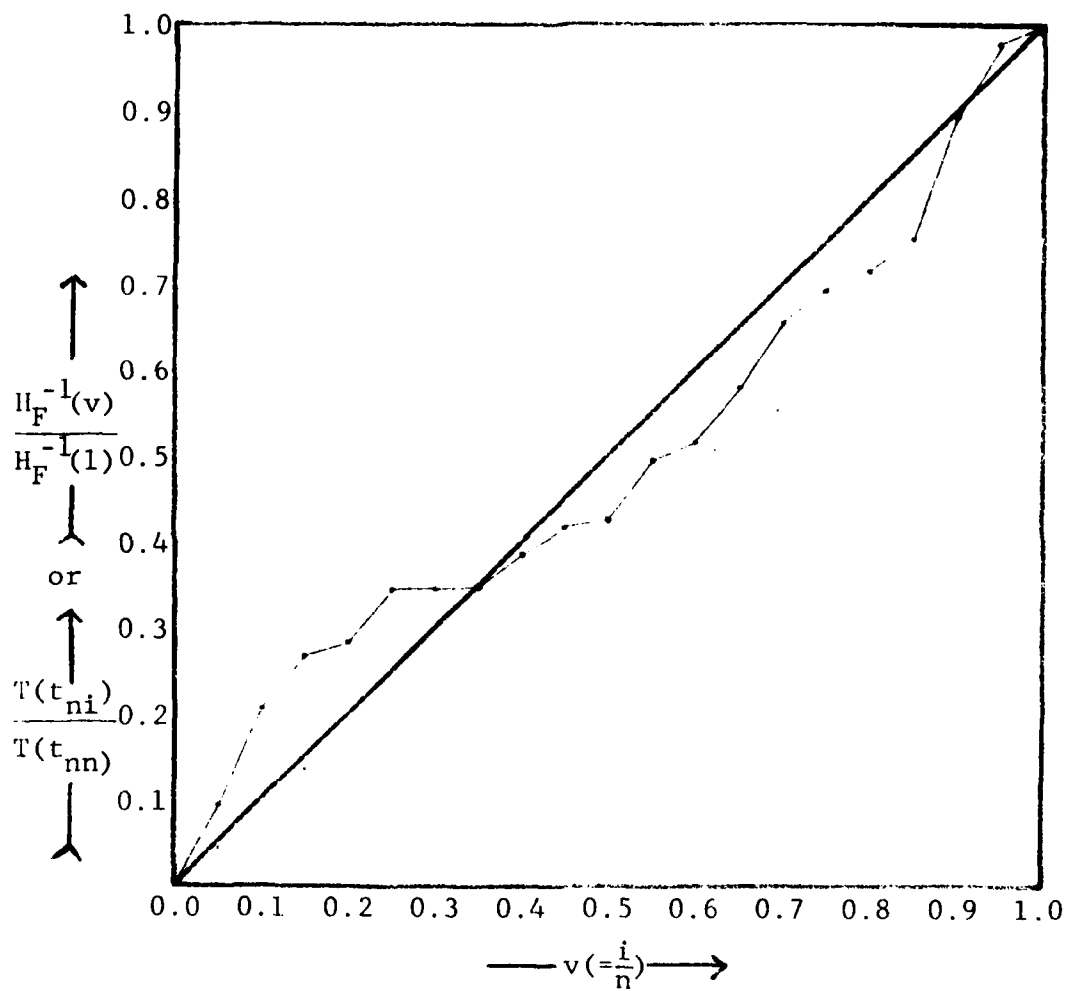
Figure 6: Scale Total Time on Test Plot for the Test Data for a Half-Ton Truck

4.2 Total Time on Test Plots with Incomplete Data

One of the important features of the total time on test plots is that we can evaluate the reliability when we have incomplete data. Incomplete data may be due to time truncated testing, censored data or the failure data containing withdrawals. In addition, the methodology can also handle grouped data. For all of these situations, we are going to present the methodology and explain how to plot the total time on test graphs. The methodology will also be explained by the help of sample problems which are typical of the military situation when testing for reliability. Test situations with incomplete data are very common and hence it is important to have a methodology to deal with this situation. There are several reasons why all the items put on test cannot be tested to failure including truncating the test due to economic reasons.

4.2.1 Total Time on Test Plots Based on Grouped Data

In many test situations, failures are recorded in terms of number of failures within specified time intervals. The actual times at which failures occur are not observed. Let us define the grid on the time axis such that failures are observed during each interval. Let the grid be

$$0 \equiv W_{0,n} < W_{1,n} < \ldots < W_{j,n} < W_{j+1,n} < \ldots \qquad (38)$$

We call the interval $[W_{j,n}, W_{j+1,n}]$ a window as this is the $(j+1)$th interval during which failures are recorded and the data

consists of total number of failures during this interval. In each window, we choose a point $\xi_{j,n}$, (perhaps the midpoint), such that $W_{j,n} \leq \xi_{j,n} < W_{j+1,n}$.

Let F_n (similar as defined by eq. (7)) be an approximation to the empirical distribution defined at $W_{j,n}$ and $\xi_{j,n}$ for $j = 1$, $2,\ldots,m$ where m is the minimum index such that all windows beyond $W_{m,n}$ have no observations.

Then, using eq. (10), we observe that the total time on test plot for grouped data is given by

$$H_n^{-1}[F_n(W_{k,n})] \overset{\Delta}{\equiv} \sum_{j=1}^{k} [1-F(\xi_{j,n})][W_{j,n} - W_{j-1,n}] \qquad (39)$$

As $n \to \infty$, we can prove [3] that the above total time on test transform for grouped data converges uniformly to $H_F^{-1}(t)$.

4.2.1.1 Example

We are interested in evaluating the reliability of a sub-system of a tank. We are evaluating the reliability based on field data and 100 subsystems were monitored for failures. We have the following data

Table 3 - Failures of a Tank Subsystem

Kilometers	Number of Failures
$t < 1000$	11
$1000 \leq t < 2000$	26
$2000 \leq t < 3000$	28
$3000 \leq t < 4000$	20
$4000 \leq t < 5000$	14
$5000 \leq t < 6000$	1
$6000 \leq t < \infty$	0
	100

28

Thus, for this example we have 6 windows or subintervals during which failures are observed. The grid is uniform because all the windows are of equal length. The approximation of the empirical distribution F_n is

$W_{k,n}$	$F_n(W_{k,n})$
1,000	0.11
2,000	0.37
3,000	0.65
4,000	0.85
5,000	0.99
6,000	1.00

Also, if we use the midpoint of each window, we have

$\xi_{k,n}$	$F_n(\xi_{k,n})$
500	0.11
1,500	0.37
2,500	0.65
3,500	0.85
4,500	0.99
5,500	1.00

Hence, using eq. (39), we can compute the following table for the total time on test transform.

$F_n(W_{k,n})$	$H_n^{-1}(F_n(W_{k,n}))$
0.11	890
0.37	1,520
0.65	1,870
0.85	2,020
0.99	2,160
1.00	2,170

We also know that $H_n^{-1}(1)$ is equal to the expected life for the tank subsystem when $n \to \infty$. Hence, the scaled total time on test plot can be developed using the following table.

v	$H_F^{-1}(v)/H_F^{-1}(1)$
0.11	0.41
0.37	0.70
0.65	0.86
0.85	0.93
0.99	0.99
1.00	1.00

The scaled total time on test plot using the data in the previous table is given in Figure 7. For this plot, it is clear that the underlying failure distribution has an increasing failure rate on average property. This distribution may also be modeled using the Weibull or gamma distribution.
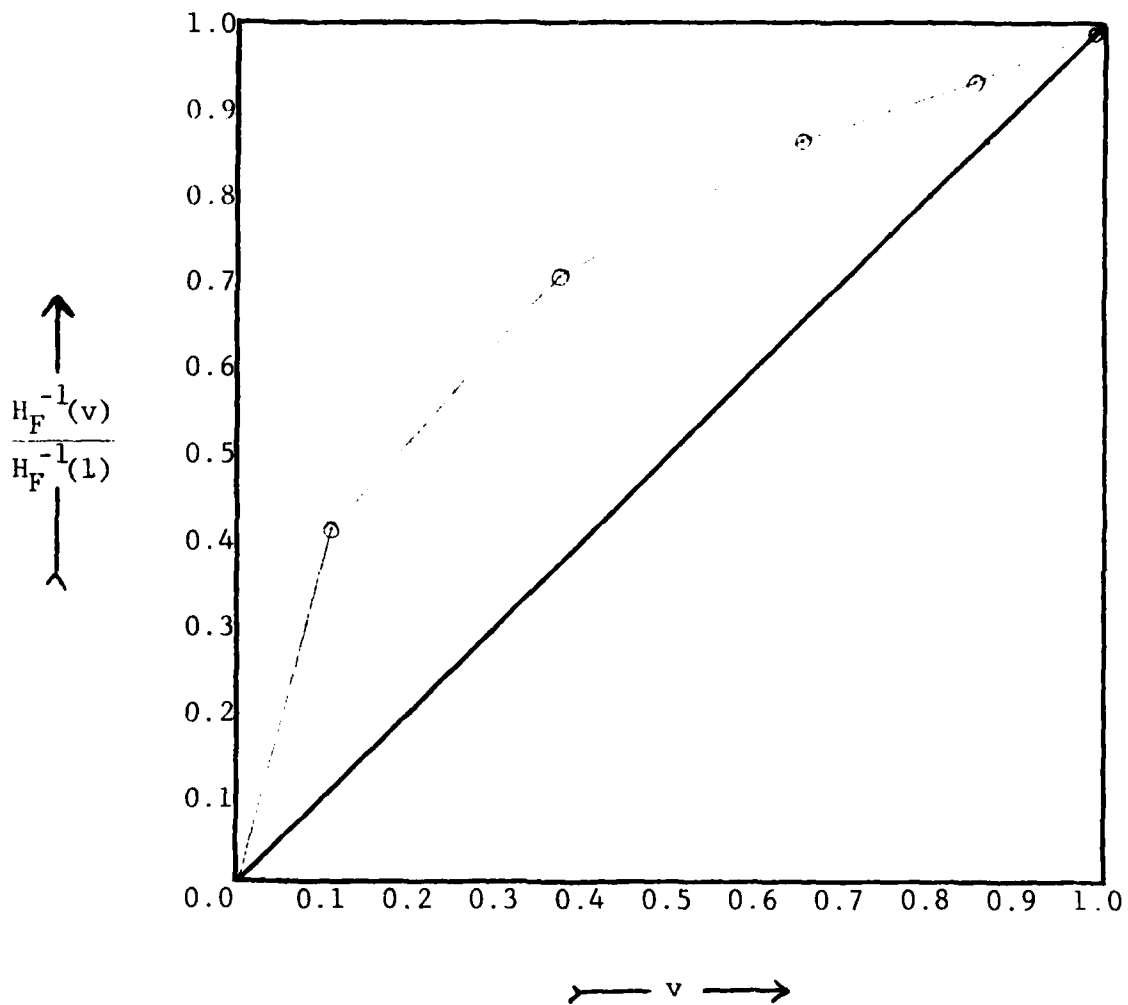
Figure 7 <u>Scaled Total Time on Test Plot for the</u>
<u>Failures of a Tank Subsystem</u>

4.3 Total Time on Test Plots with Truncated Data

For several test situations, testing is done for a given period and is terminated at some point in time T_{max}. If we had n items on test, then we observe $r \leq n$ failures over the interval $[0, T_{max}]$. Thus, the total time on test for this situation is (see eq. (34))

$$T(t) = \int_0^t n(u) \, du \quad , \quad 0 \leq t \leq T_{max} \tag{40}$$

For this situation we plot $T(t_{ni})/T(T_{max})$ versus $\frac{i}{k}$, $i = 1, 2, \ldots, k$. For this we have

$$\frac{T(t_{ni})}{T(T_{max})} \to \frac{H_F^{-1}(vp)}{H_F^{-1}(p)} \tag{41}$$

as $n \to \infty$ and $\frac{i}{k} \to v$ where $p = F(T_{max})$. Hence, total time on test data plots should be compared with suitably scaled total time on test transforms.

4.4 Total Time on Test Plots with Censored Data

In many test situations, we place n items on test and stop the test as soon as the rth (r n) failure is observed. For this situation we plot

$$\frac{T(t_{ni})}{T(t_{nr})} \quad \text{versus} \quad \frac{i}{r} \ , \quad i = 1,2,\ldots,r \tag{42}$$

The underlying limiting convergence for this case is

$$\frac{T(t_{ni})}{T(t_{nr})} \rightarrow \frac{H_F^{-1}(vp)}{H_F^{-1}(p)} \tag{43}$$

as $n \rightarrow \infty$, $\frac{r}{n} \rightarrow p$ and $0 \leq v \leq 1$.

Thus, again the total time on test data plots should be compared with suitably scaled total time on test transforms.

4.5 Total Time on Test Plots for Failure Data Containing Withdrawals or Suspended Items

In some test situations, items are taken off of test for reasons other than failure [14]. For instance, a test may be stopped if a test stand breaks down or a test vehicle becomes involved in an accident. Or, we may want to purposely place more items on test than we intend to fail in order to decrease the test time. In this section, we are presenting a methodology to handle suspended items or withdrawals using the concept of total time on test plots.

Let t_1, t_2, ..., t_n be independent random variables with distribution F. Suppose, we only observe

$$Y_i = \min (t_i, \ell_i) \quad i = 1, 2, ..., n \tag{44}$$

where ℓ_i's are constants. If $t_i > \ell_i$, then we say that item i was lost to observation at time ℓ_i or the item was suspended at time ℓ_i. Hence, let

$$0 \equiv t_{n0} \leq t_{n1} \leq \cdots \leq t_{nk} \tag{45}$$

be the observed failure times. For this case, the total time on test is given by (using eq. (34))

$$T(t_{ni}) = \int_o^{t_{ni}} n(u) \, du \tag{46}$$

Hence, we plot

$$\frac{T(t_{ni})}{T(t_{nk})} \quad \text{versus} \quad \frac{i}{k} \, , \quad i = 1, ..., k \tag{47}$$

34

For this situation, the model identification is not possible using the scaled total time on test plots for well known distributions for comparison. However, we can show that the total time on test plot will tend to lie above (below) the 45° line if F is IFR(DFR).

4.5.1 Example

Let us consider an example of failure data with suspended items as given in reference [14]. We put 10 items on test but only observe six failures and four items are suspended or withdrawn during testing. The test data is given in Table 4.

Based on this table, the ordered six failure times are

544 < 663 < 897 < 914 < 1084 < 1099

Hence, the total time on test are given in Table 5. The table has been developed using Figure 8.

Table 4 - Suspended Test Data

Hours on Test	Status of Item
544	Failure
663	Failure
802	Suspension
827	Suspension
897	Failure
914	Failure
939	Suspension
1084	Failure
1099	Failure
1202	Suspension

Table 5 - <u>Total Time on Test</u>

i	t_{ni}	$T(t_{ni})$	$T(t_{ni})/T(t_{nk})$	i/k
1	544	5440	0.613	0.166
2	663	6511	0.734	0.333
3	897	8218	0.926	0.500
4	914	8303	0.936	0.666
5	1084	8838	0.996	0.833
6	1099	8868	1.000	1.000

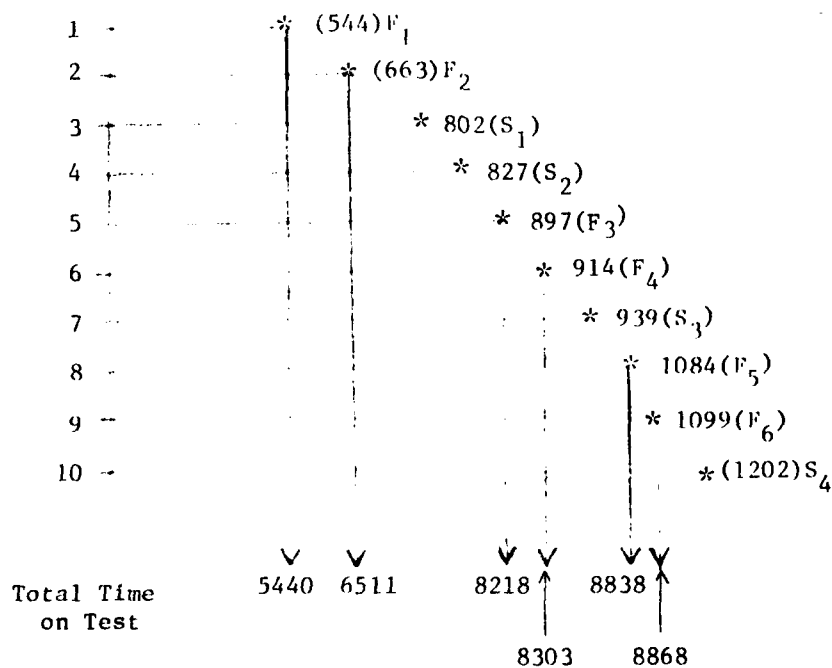Figure 8 - <u>Total Time on Test Computation for</u>

<u>the Suspended Test Data</u>

30

The data given in Table 5 is plotted in Figure 9. This is the scaled total time on test plot for the failure data with suspended items. Based on this plot, we can say that the underlying failure distribution has increasing failure rate (IFR).
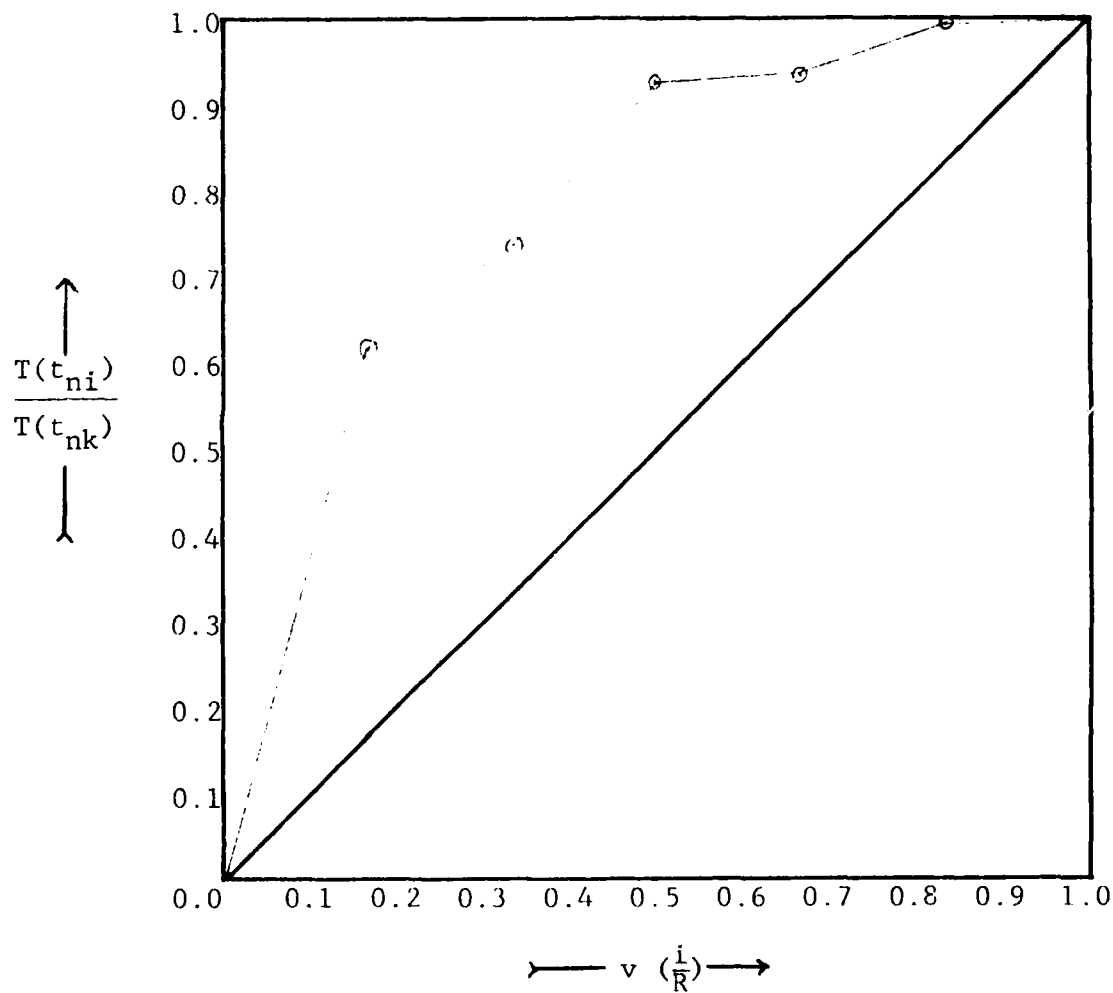
Figure 9   Scaled Total Time on Test Plot for Suspended
           Test Data

5.0 Some Important Results on Total Time on Test Plots

Let the observed failure times with or without censoring be

$$0 = t_{n0} \leq t_{n1} < \cdots < t_{nk} \tag{48}$$

as given in section 4.5. If the underlying failure distribution is exponential as given by

$$F(t) = 1 - e^{-t/\theta} \tag{49}$$

then the total time on test until the next failure is given by

$$\int_0^{t_{n1}} n(u) \, du \quad , \quad \int_{t_{n1}}^{t_{n2}} n(u) \, du, \quad \cdots \int_{t_{n,k-1}}^{t_{nk}} n(u) \, du \tag{50}$$

and these are distributed as independent random variables each with the exponential distribution. Let us make an assumption about the test procedure such that we observe at least k failures when n items were put on test at time t = 0. Let us say that the unit i has age a_i when testing commences. It is withdrawn at time $\ell_i > a_i$ if it does not fail in the interval (a_i, ℓ_i). Let $\ell_i = \infty$, i = 1, 2,...,k ($1 \leq k \leq n$) so that we observe at least k failures. Under this assumption, the random variables given by eq. (50) are independent and exponentially distributed. Let us define

$$U_{k-1,i} \triangleq \frac{T(t_{ni})}{T(t_{nk})} \qquad i = 1, 2, \ldots, k-1 \tag{51}$$

Then $U_{k-1,i}$ are distributed as (k-1) order statistics from a uniform distribution on [0, 1]. Hence,

$$E[U_{k-1,i}] = \frac{i}{k} \qquad i = 0, 1, 2, \ldots, k-1 \qquad (52)$$

and sample plots tend to follow the 45° line on the scaled total time on test graphs. Since

$$P[U_{n-1,i} < \frac{1}{n}] = 1 - (1 - \frac{1}{n})^{n-1}$$

$$\approx 1 - e^{-1} \qquad (53)$$

we see that, under exponentiality, the plot is likely to lie with probability $1 - e^{-1} \approx 0.632$ initially below the 45° line. If we have infant mortality this means initially a DFR distribution, and we would of course expect the plot to initially lie below the 45°. Now, we state an important theorem for the exponential distribution.

5.1 Theorem

If F is exponential and we observe n failures during the test, then

P {Total Time on Test Plot lies above 45° line}

= P {Total Time on Test Plot lies below 45° line}

$$= \frac{1}{n} \tag{54}$$

Let us compute the probabilities of the following events associated with the behavior of the total time on test plots. Let

E_1: The plot is initially below and finally below the 45° line.

E_2: The plot is initially below and finally above the 45° line.

E_3: The plot is initially above and finally below the 45° line.

E_4: The plot is initially above and finally above the 45° line.

Then we have from the joint density of U_{n1} and $U_{n,n-1}$

$$P(E_1) = (1 - \frac{1}{n})^{n-1} - (1 - \frac{2}{n})^{n-1} \approx 0.23$$

$$P(E_2) = 1 - 2(1 - \frac{1}{n})^{n-1} + (1 - \frac{2}{n})^{n-1} \approx 0.4$$

$$P(E_3) = (1 - \frac{2}{n})^{n-1} - (1 - \frac{2}{n})^{n-1} \approx 0.14$$

$$P(E_4) = (1 - \frac{1}{n})^{n-1} - (1 - \frac{2}{n})^{n-1} \approx 0.23$$

Thus, it is clear that the event E_2 is the most likely of the four events considered above.

5.2 Concept of Stochastic Ordering and Its Applications to Total Time on Test Plots

It is important to provide stochastic comparisons of total time on test processes for various classes of distributions.

A stochastic process $\{X(t), 0 < t < 1\}$ stochastically dominates $(\overset{st}{\geq})$ a process $\{Y(t), 0 \leq t \leq 1\}$ if

$$f[\{X(t), 0 \leq t \leq 1\}] \overset{st}{\geq} f[\{Y(t), 0 \leq t \leq 1\}] \qquad (55)$$

for every increasing functional f. The symbol $(\overset{st}{\geq})$ means stochastic or probability ordering.

Based on the concept of stochastic ordering we have the following result which can be used for stochastic ordering of total time on test plots.

If the underlying failure distribution is IFRA (DFRA), then the scaled total time on test plot stochastically dominates (is dominated by) corresponding plots based on an exponential distribution of the same sample size. We stated before that $H_F^{-1}(t)/H_F^{-1}(1)$ lies above (below) the 45° line when F is IFRA (DFRA). Thus, if the test data plot lies above the 45° line we should reject an exponential model in favor of an IFRA (or IFR) model.

42

5.3 Analysis of Censored Data or Data with Suspended Items

Model identification for censored data is very difficult and theoretical results are not available at the present time. However, we can still test for exponentiality versus alternative models.

It is clear that $F_T \underset{*}{<} F$ for all continuous F, where

$$F_T(x) = \begin{cases} F(x) , & 0 \le x < T \\ 1 , & x \ge T \end{cases} \tag{56}$$

Thus the scaled total time on test plots from censored data always stochastically dominated scaled total time on test plots from complete data having the same sample size. Similarly, $F_p \underset{*}{<} F$, where

$$F_p(x) = \begin{cases} \dfrac{1}{p} F(x) , & 0 < x \le F^{-1}(p) \\ 1 , & x > F^{-1}(p) \end{cases}$$

and $p = \dfrac{k}{n}$, so that again the total time on test plots when observations are terminated at the k th failure dominate scaled plots from complete data having the same sample size.

Many times we have test data where items are suspended. Thus, the test data indicates that the i th item survived up to time ℓ_i, and after that it was suspended. One can again model this situation using the concept of stochastic ordering and prove that the scaled total time on test plot for suspended items tends to lie above the scaled total time on test plot without suspended items.

43

6.0 Computation of Failure Rate & Mission Reliability

In Section 2.1, we proved that

$$\frac{d}{dv} \left. H_F^{-1}(v) \right|_{v = F(t)} = \frac{1}{h(t)} \qquad (58)$$

which is the relationship between failure rate $h(t)$ and $H_F^{-1}(v)$. In order to compute empirically the failure rate, we consider the case when the scaled total time on test plots are joined by straight lines as shown in Figures 7 and 9. Let $0 = u_0 < u_1 < \ldots < u_k < 1$ be the abscissa values of endpoints of linear segments. Then the successive slopes of the linear segments denoted by $a_1, a_2, \ldots, a_{k+1}$ is given by

$$a_i = \frac{[H_n^{-1}(u_i) - H_n^{-1}(u_{i-1})]}{(u_i - u_{i-1}) \, H_n^{-1}(1)} \qquad (59)$$

And, using eq. (58), it is clear that the successive distribution failure rates are given by

$$\lambda_i = \frac{H_n^{-1}(1)}{a_i} \qquad (60)$$

where λ_i is the failure rate for the ith linear segment. Thus, we assume that we have a constant failure rate λ_i in the interval $t_{i-1} \leq t < t_i$, $i = 0, 1, \ldots, k$. In order to compute t_i, we can approximate the underlying distribution by a piecewise exponential distribution with different failure rates. The cumulative failure rate $\lfloor \lambda(t_r) \rfloor$ estimate at the time t_t is given by

44

$$\lambda(t_r) = \sum_{i=1}^{r} \lambda_i (t_i - t_{i-1}) \qquad (61)$$

Hence,

$$u_r = F(t_r) = 1 - \exp[-\sum_{i=1}^{r} \lambda_i (t_i - t_{i-1})] \qquad (62)$$

Thus, t_i's can be recursively computed. We have

$$t_1 = -[\log (1-u_1)]/\lambda_1 \qquad (63)$$

and in general

$$t_r = [-\log(1-u_r) - \sum_{i=1}^{r-1} \lambda_i (t_i - t_{i-1})]/\lambda_r + t_{r-1} \qquad (64)$$

After we have computed the failure rate λ_i for the interval $t_{i-1} \leq t < t_i$, we can compute the mission reliability for any mission of duration T by observing that

Mission Reliability = R(T)

$$= \exp[-\int_0^T h(u) \, du] \qquad (65)$$

Hence, eq. (65) can be approximated by

$$R(T) = \exp[-\sum_{i=1}^{r} \lambda_i (t_i - t_{i-1}) - \lambda_{r+1}(T - t_r)] \qquad (66)$$

where T is such that $t_r \leq T < t_{r+1}$.

SECTION II

THE EXPONENTIAL DISTRIBUTION AND

MISSION RELIABILITY ESTIMATION

THE EXPONENTIAL DISTRIBUTION AND
MISSION RELIABILITY ESTIMATION

1.0 Introduction

The exponential distribution is an extremely popular and easy to use distribution. It is applicable during the useful life portion of a vehicle system when the failure rate can be assumed to be constant. Thus, it would not be applicable for extended predictions concerning vehicle durability.

2.0 The Basic Statistical Model

The p.d.f. for an exponentially distributed random variable t is given by

$$f(t) = \frac{1}{\theta} e^{-t/\theta} \quad , \quad t \geq 0 \tag{1}$$

In typical vehicle testing the quantity t, would represent the miles (or kilometers) between failures (MMBF). The parameter θ is the mean miles between failure. The failure rate is given by λ where

$$\lambda = 1/\theta \tag{2}$$

The shape of the exponential p.d.f. is shown in Figure 1. In this figure R(t) is the reliability. Analytically the reliability function is calculated by

$$R(t) = e^{-t/\theta} \quad , \quad t \geq 0 \tag{3}$$

Or, using the failure rate

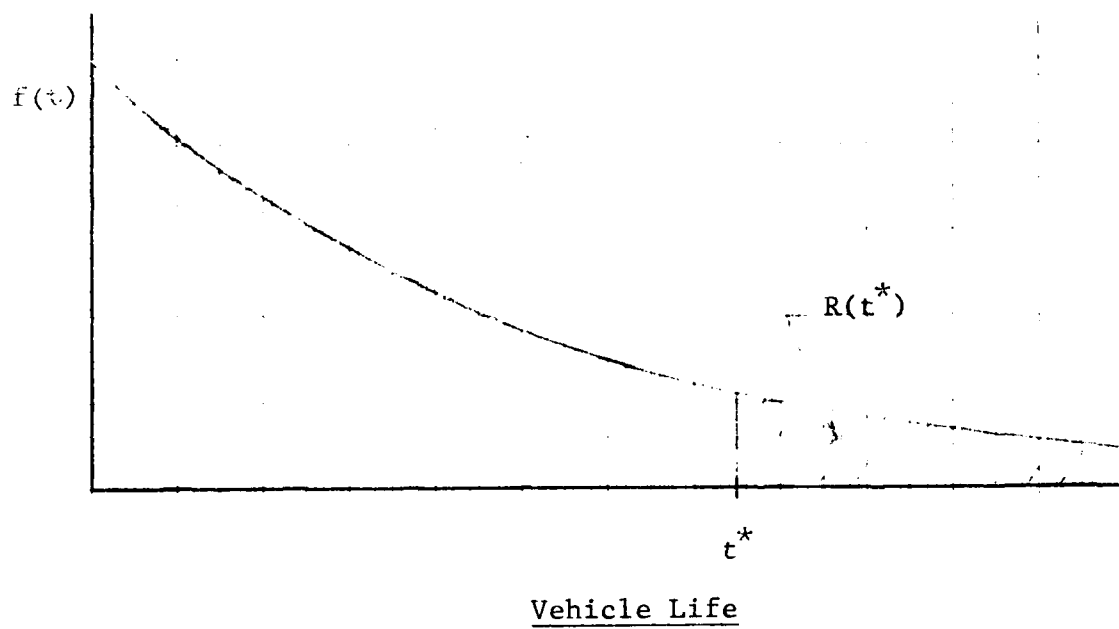$$R(t) = e^{-\lambda t} \quad , \quad t \geq 0 \tag{4}$$

Figure 1   The Exponential Life Distribution (p.d.f.)

The MMBF for the exponential distribution is given by the parameter θ. The chances of surviving this time can be found from equation 3. Substituting into this equation gives

$$R(\theta) = e^{-1} = 0.368 \tag{5}$$

Or, there is only a 37% chance of surviving the mean life when the underlying distribution is exponential.

EXAMPLE 1

The mission length for a 5-ton truck is specified as 75 miles. The MMBF is 2,150 miles. Let us find the mission reliability.

Using equation 3 with $\theta = 2{,}150$ miles we find

$$R(75 \text{ miles}) = e^{-75/2,150} = 0.97$$

which is the mission reliability.

EXAMPLE 2

The required mission reliability for an armored personnel carrier is 0.92 with a mission length of 50 miles. Assuming an exponential time to failure distribution let's find the required vehicle MMBF.

Using equation 3 we can find that

$$\theta = t_m / \ln (1/R_m) \tag{6}$$

where t_m is the required mission length and R_m is the required mission reliability. Substituting into equation 6 we find

$$\theta = 50 \text{ miles}/\ln\ (1/0.92) = 600 \text{ miles}$$

this means that a 600 mile MMBF is required to achieve the desired reliability goal. Or, saying this another way, the mission failure rate can be no greater than

$$\lambda = 1/600 = 1.67 \times 10^{-3} \text{ failures/miles}$$

2.1 Goal Setting Using Mission Reliability

Goal setting using mission reliability rather than vehicle MMBF can be dangerous if one is not familiar with the consequences. Figure 2 illustrates the percent improvement needed to accomplish a given improvement in mission reliability. This trade off is dependent on the initial reliability level of the vehicle.

As an example of using Figure 2 say we want a 5% improvement in mission reliability. If the initial vehicle mission reliability was 80% the required increase in vehicle MMBF would be 28.5%. It should be recognized that such a large change in vehicle MMBF is usually extremely difficult to accomplish. Note also that if the vehicle originally had a 0.90 mission reliability the required increase in vehicle MMBF would be over 50%, which is impossible to accomplish in most PIP programs.

2.2 Estimating Mission Reliability

The duration of testing for each vehicle is normally specified by the vehicle developer. Testing is then carried out at a designated proving ground. In life testing terminology this is Type II (or time) censored life testing.
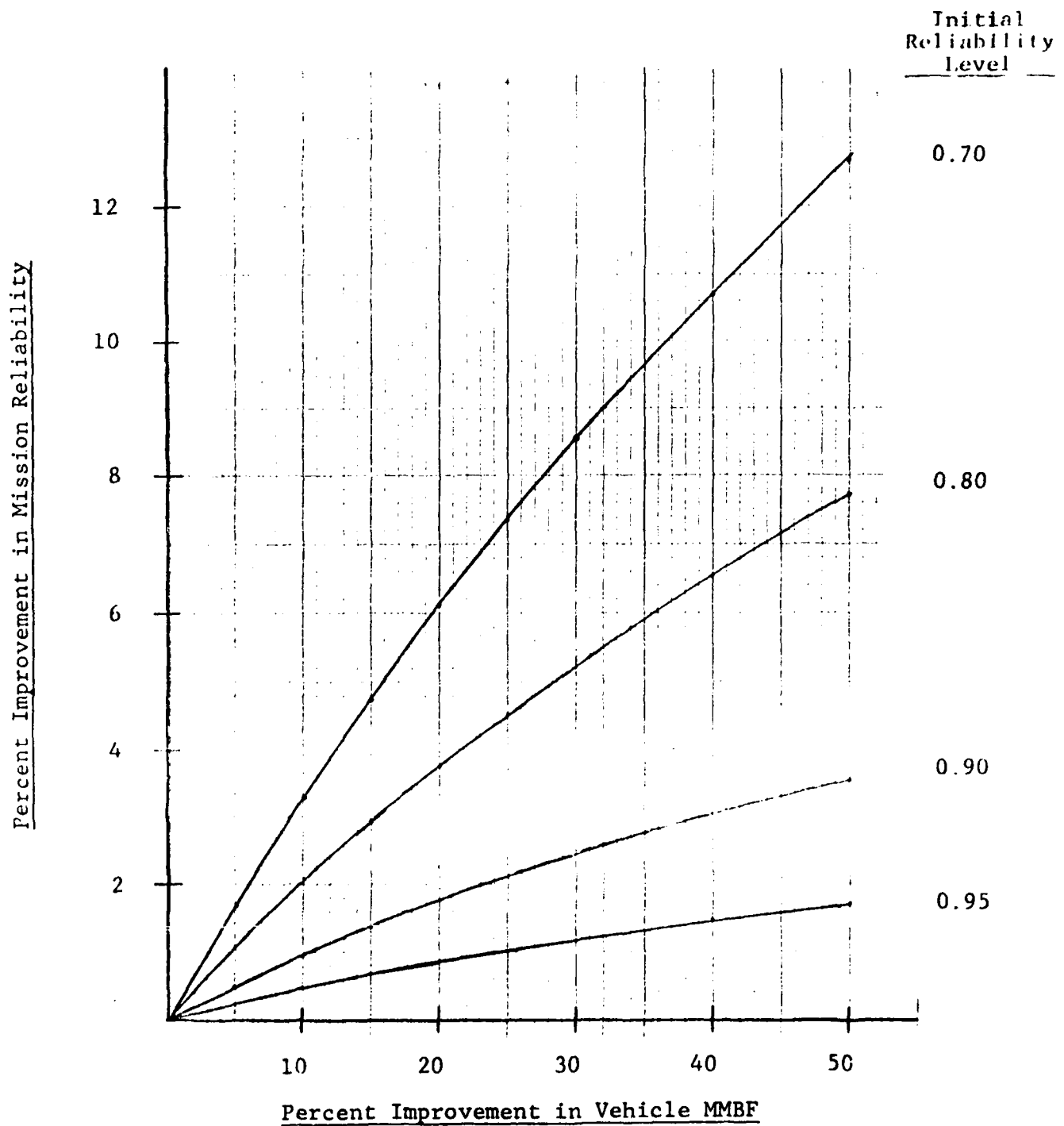
Figure 2: Mission Reliability Improvement as a Function of Improvement in Vehicle MMBF

2.2.1 Point Estimate: The vehicle MMBF is estimated by

$$\hat{\theta} = \frac{\text{Total accumulated test miles}}{\text{Total number of scored failures}} = \frac{T}{r} \qquad (7)$$

The mission reliability is then estimated as

$$\hat{R}_m = e^{-t_m/\hat{\theta}} \qquad (8)$$

where t_m is the mission length.

2.2.2 Confidence Limit Estimate: The lower $100(1-\alpha)\%$ confidence limit on the vehicle MMBF is given by

$$\frac{2T}{\chi^2_{\alpha,\ 2r+2}} \leq \theta \qquad (9)$$

where T is the total accumulated test time and r is the total number of scored failures. The quantity $\chi^2_{\alpha,\ 2r+2}$ is obtained from a chi-square table using 2r+2 as the degrees of freedom.

A convenient approximation based on the normal distribution is given by

$$\frac{4T}{[z_\alpha + \sqrt{4r+3}]^2} \leq \theta \qquad (10)$$

where z_α is a standard normal variate [1,14]. This approximation is fairly good with $r \geq 5$ and gets better with increasing r. Common values for z_α are listed below.

Table I: Standard Normal Variates

Confidence Level $(1-\alpha)$	Confidence Factor (z_α)
95%	1.645
90%	1.281
80%	0.841
70%	0.525
50%	0.0

A convenient way to quickly approximate a 50% confidence limit is given by

$$\frac{T}{r+0.75} \leq \theta \qquad (11)$$

where r is the number of failures.

The $100(1-\alpha)\%$ lower confidence limit on mission reliability is given by

$$e^{-t_m/L} \leq R_m \qquad (12)$$

where L is the $100(1-\alpha)\%$ lower confidence limit on the MMBF as calculated from equations 9, 10, or 11. Of course t_m is the mission length.

A 50% lower confidence limit on mission reliability is conveniently approximated by

$$e^{-t_m(r+0.75)/T} \leq R_m \qquad (13)$$

EXAMPLE 3

Three XM-1 tanks were each scheduled for 4,000 miles of testing. The test produced 26 scored mission failures. Let's estimate the mission reliability.

Using equation 7 the vehicle MMBF is estimated as

$$\hat{\theta} = \frac{3 \text{ tanks X } 4,000 \text{ mi/tank}}{26 \text{ failures}} = 462 \text{ miles} \quad .$$

Assuming a mission length of t_m = 100 miles, the point estimate of mission reliability is obtained from equation 8 and is

$$\hat{R}_m = e^{-11/462} = 0.81$$

The 50% confidence limit is calculated from equation 13 and is

$$e^{-100(26+0.75)/12,000} \leq R_m$$

$$0.80 \leq R_m$$

2.3 Sequential Testing

Sequential testing allows one to continually assess vehicle acceptance on a day by day basis. As soon as an accept (or reject) decision is made the test is terminated. This method minimizes testing and is more economical and fuel efficient. However, sequential acceptance tests are slightly harder to design.
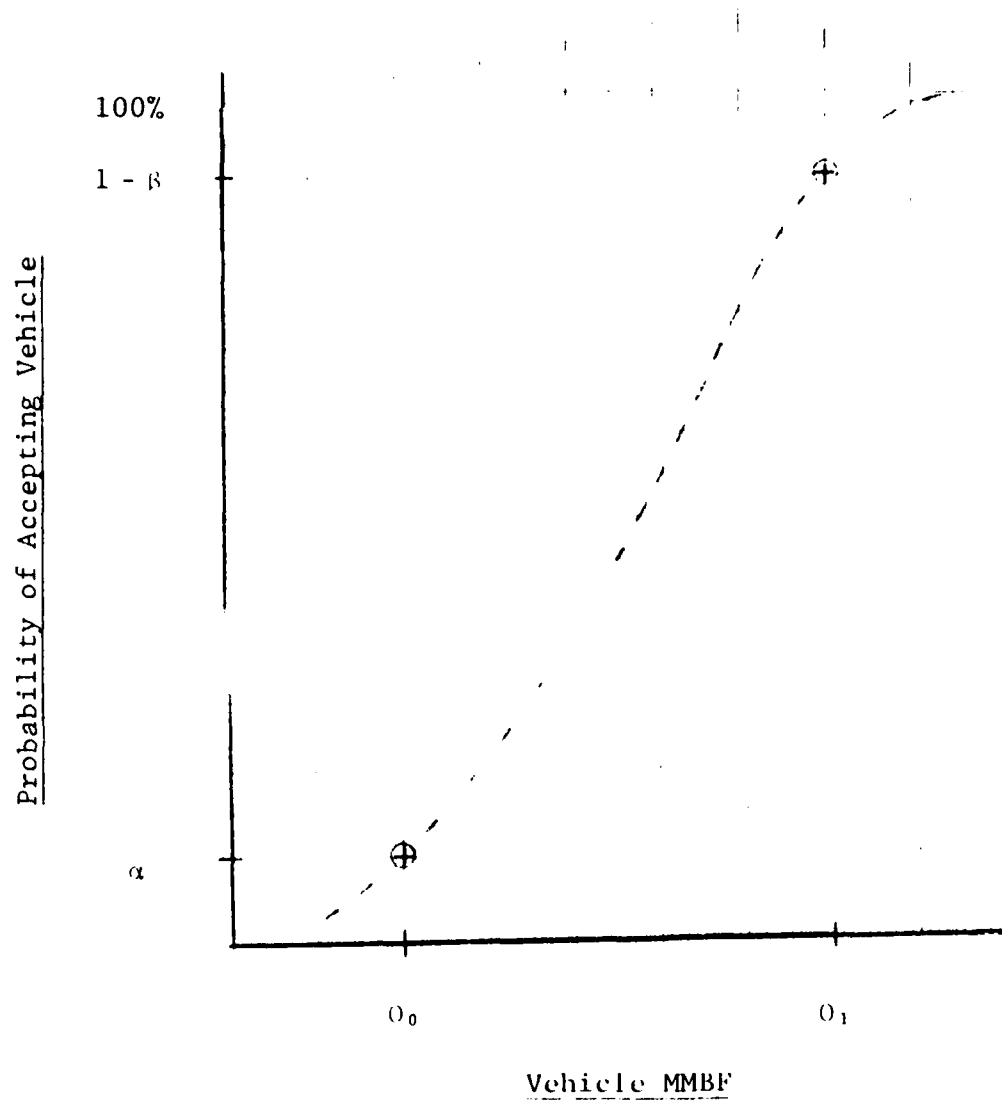
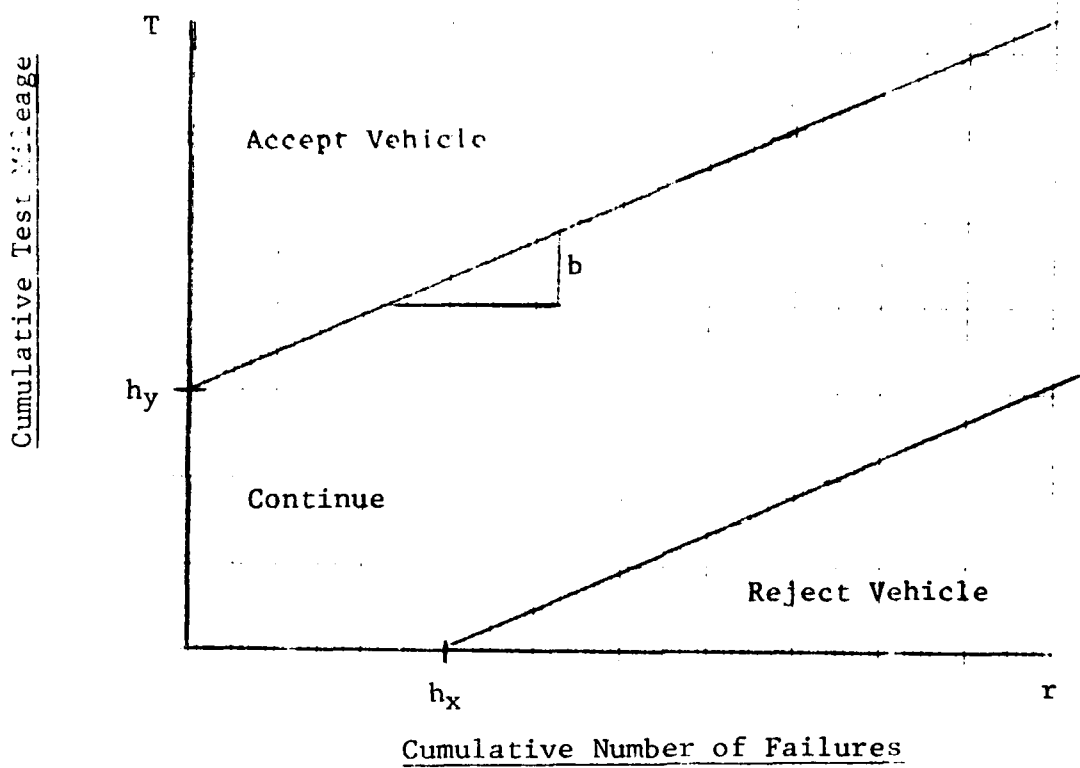Figure 3 Operating Characteristic Curve for a Sequential Test

Figure 4 Sequential Test Graph

The parameter that determine a sequential life test are ϵ, θ_0, and ϵ, θ_1. The relationship between these parameters is shown in Figure 3. As can be seen in this figure these points determine the Operating Characteristic (O.C.) curve for the test. Intermediate points on the O.C. curve can be calculated from the equation in Table II with $i = 0.1$ and/or 0.5.

Table II: Calculation of Sequential O.C. Curve

True Value of MMBF	Probability of Accepting Vehicle
θ_0	α
$\theta_i = \dfrac{1-(\frac{\theta_0}{\theta_1})^i}{i(\frac{\theta_1-\theta_0}{\theta_1\theta_0})}$	$(1-A^i)/(B^i-A^i)$
θ_1	$1 - \beta$

$$A = \frac{\beta}{1-\alpha}$$

$$B = \frac{1-\beta}{\alpha}$$

Once the O.C. curve is decided on, the test is determined. In order to set up the test boundaries the following quantities must be calculated.

$$b = [\frac{\theta_1 - \theta_0}{\theta_0 \theta_1}] \frac{1}{\ln(\theta_1/\theta_0)} \tag{14}$$

$$h_y = \left[\frac{\theta_0 \theta_1}{\theta_1 - \theta_0}\right] \ln\left(\frac{1-\beta}{\alpha}\right) \tag{15}$$

$$h_x = \frac{\ln\left(\frac{1-\alpha}{\beta}\right)}{\ln(\theta_1/\theta_0)} \tag{16}$$

These quantities are used to draw the sequential life testing graph as shown in Figure 4. Both lines are parallel with slope b. This graph allows monitoring of the test on a day to day basis showing the results visually.

EXAMPLE 4

A 10-ton Hi-Mo vehicle must demonstrate an MMBF of 2,500 miles in an IP test. It is decided to use a sequential test in an effort to minimize fuel consumption by minimizing testing.

In order to design the test the following parameters are selected

$$\beta = 0.20$$
$$\theta_1 = 2,500 \text{ miles}$$
$$\alpha = 0.10$$
$$\theta_0 = 2,000 \text{ miles}$$

By using Table II with i = 0.5 an intermediate point on the O.C. curve is calculated as
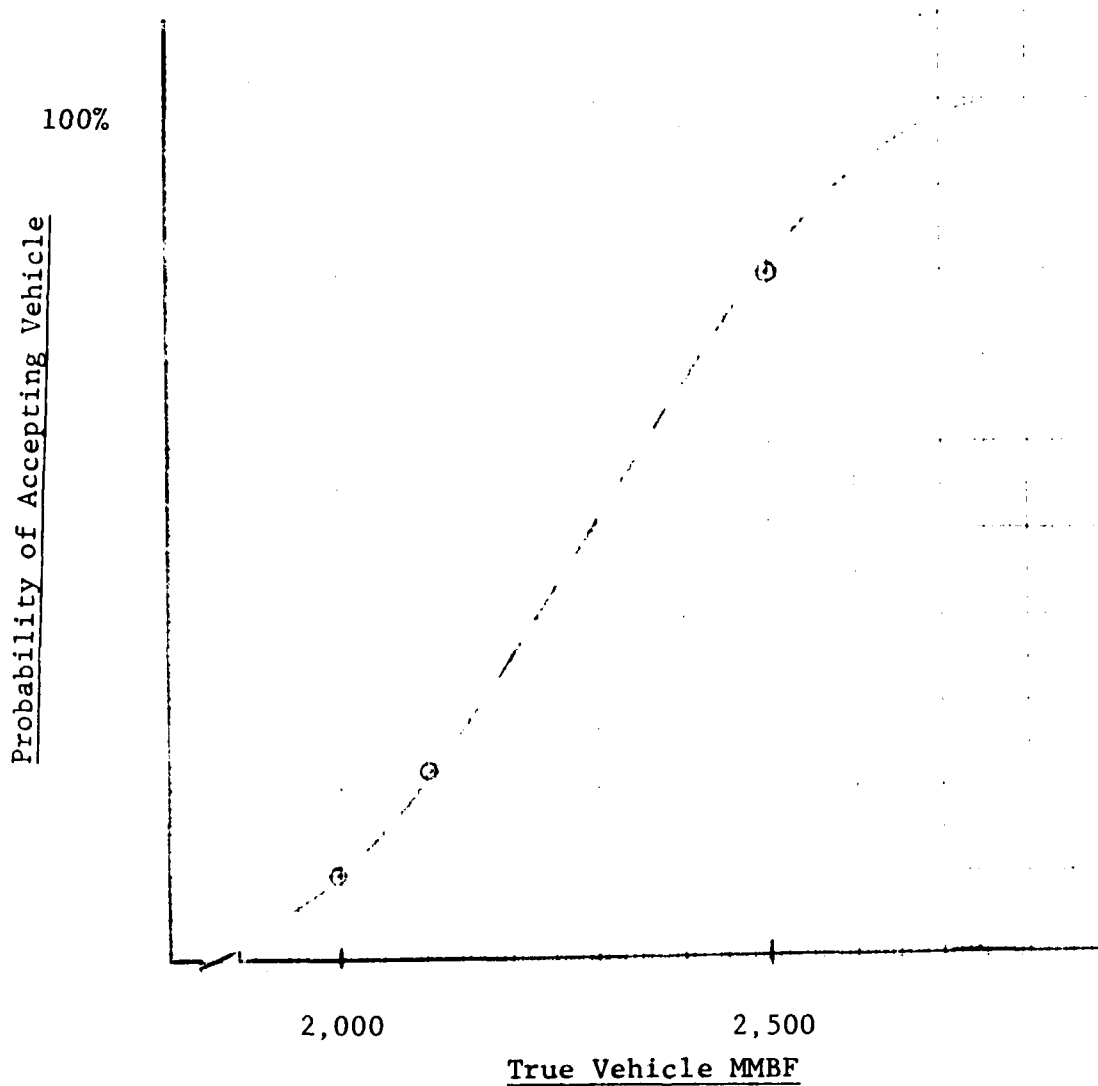
$$\theta_{0.5} = 2,111$$

Figure 5   O.C. Curve for a Vehicle Acceptance Test

with a probability of acceptance of 0.22. The resulting O.C. curve is sketched in as shown in Figure 5. If one does not like this O.C. curve then the parameters must be changed.

The quantities for the sequential test graph are found from equations 14, 15, and 16.

In this case,

$$b = 4.48 \times 10^{-4} \text{ failures/mile}$$

$$h_y = 20{,}794$$

$$h_x = 6.740$$

These factors can now be used to construct the sequential graph. This is easier to do if the slope is expressed as

$$b = \frac{4.48 \text{ failures}}{10{,}000 \text{ miles}} \qquad (17)$$

The points h_x and h_y are scaled off on the x and y axes as shown in Figure 6. Straight lines are then drawn through these points using the slope scale as determined from equation 17.
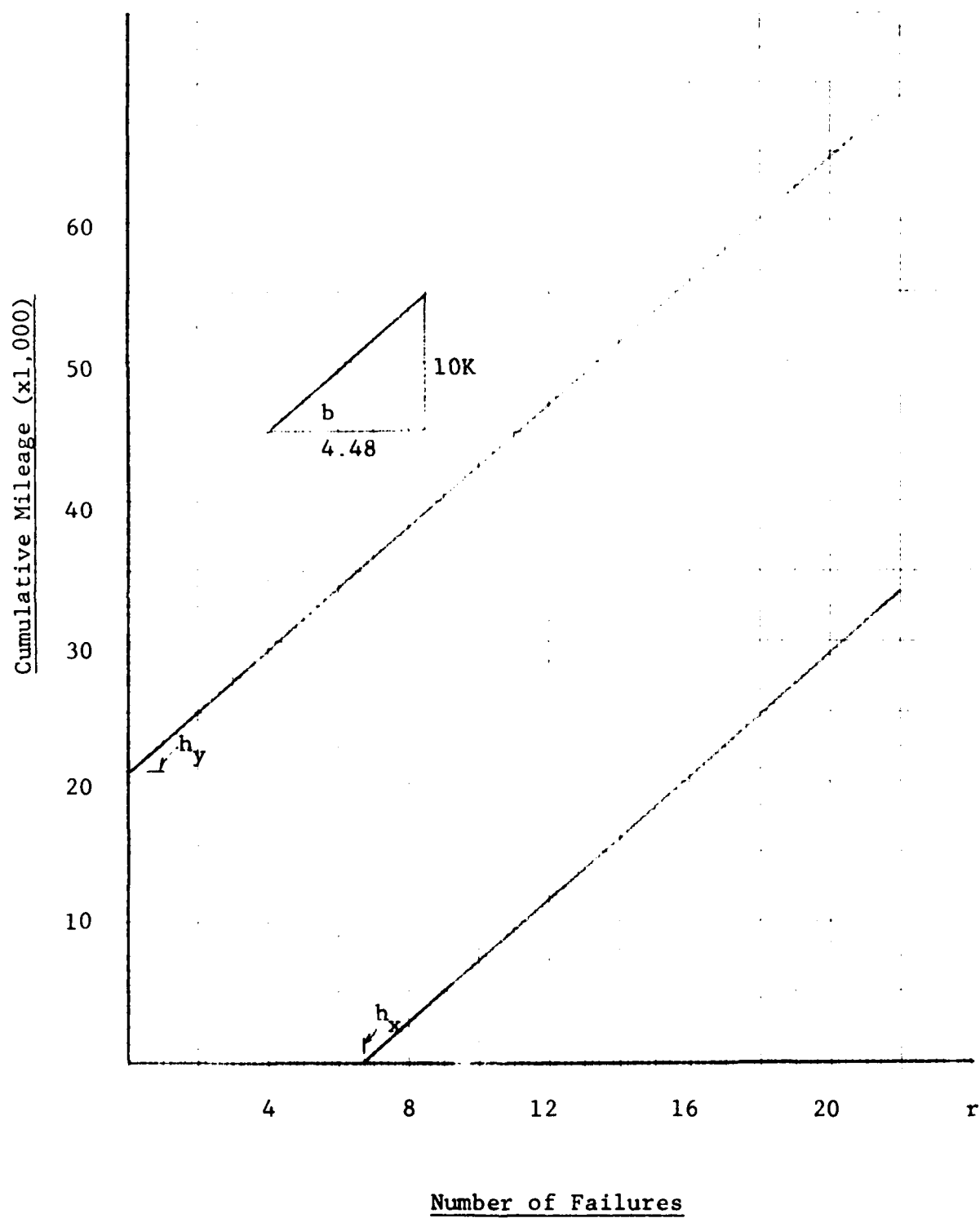
Figure 6   Sequential Testing Graph

61

SECTION III

THE BINOMIAL DISTRIBUTION AND

MISSION RELIABILITY ESTIMATION

THE BINOMIAL DISTRIBUTION AND
MISSION RELIABILITY ESTIMATION

1.0 Introduction

A vehicle mission is a precisely defined quantity that specifies such things as ground terrain, vehicle speeds, weapon usage, etc. The test profile can be looked at as a series of vehicle missions. During testing, the vehicle either succeeds or fails each mission. When the test is viewed from this standpoint, the binomial distribution can be used to determine mission reliability.

2.0 Basic Statistical Model

The probability model for this situation is

$$p(y) = \binom{n}{y} R_m^y (1-R_m)^{n-y} , \quad y = 0, 1, 2,\ldots,n \quad (1)$$

where

R_m = *mission reliability*

n = total number of missions (completed and attempted)

y = total number of missions successfully completed

The quantity $p(y)$ is the probability of completing y missions successfully out of n attempts. This is the well known binomial distribution.

Example 1

The mission reliability for an armored personnel carries is R_m 0.80. If 5 vehicles are sent on a mission what is the probability that at least 3 will succeed?

Here we want

$$p(y \geq 3) = p(y = 3) + p(y = 4) + p(y = 5)$$

which is the probability of three or more successes.

In this case,

$$p(y) = \binom{5}{y} \; 0.8^y \; 0.2^{5-y}$$

and for example

$$p(y = 3) = \frac{5!}{3!\,2!} \; 0.8^3 \; 0.2^2 = 0.205$$

Performing similar calculations gives

$$p(y \geq 3) = 0.943$$

Or, there is a 94.3% chance that at least three vehicles will complete the mission.

3.0 Mission Reliability Using the Binomial Distribution

3.1 Estimating Mission Reliability

For this testing situation both point and confidence interval estimates of mission reliability can be obtained. The basic input data is n, the number of missions attempted and y, the number of missions successfully completed.

3.1.1 Point Estimate: The point estimate of mission reliability is obtained by:

$$\hat{R}_m = \frac{y}{n} \qquad (2)$$

3.1.2 Confidence Limit Estimate: An exact $100(1-\alpha)\%$ lower confidence limit on the reliability is given by

$$R_L = \frac{y}{y + (n-y+1) \, F_{\alpha, 2(n-y+1), 2y}} \qquad (3)$$

where $F_{\alpha, 2(n-v+1), 2y}$ is easily obtained from F-tables [16]. Also recall that,

$$n = \text{total number of missions}$$

$$y = \text{number of successful missions}$$

The F-tables that are usually available are somewhat limited. Therefore, it is convenient to have an approximation for the confidence limit that uses the standard normal distribution. The lower confidence on mission reliability can be approximated by

$$R_L = \frac{(y-1)}{n + z_\alpha \sqrt{\frac{n(n-y+1)}{(y-2)}}} \qquad (4)$$

z_α = the standard normal variate as given in Table I, below.

<u>Confidence Level (1-α)</u>	<u>Confidence Factor (z_α)</u>
95%	1.645
90%	1.281
80%	0.841
70%	0.525
50%	0.0

Frequently 50% confidence limits are used in reliability work. Note that the 50% lower confidence limit is approximated by

$$R_L = \frac{y-1}{n} \tag{5}$$

Example 2

A TOW weapon system has completed a test schedule. The test is equivalent to 60 missions. Dividing the test schedule up into 60 missions results in seven failed missions. Let's estimate the mission reliability.

$y = 60 - 7 = 53$ successful missions

out of

$n = 60$ missions

Then the point estimate for mission reliability is

$$\hat{R}_m = \frac{53}{60} = 0.883$$

The lower 75% confidence level is found from equation 3, with

$$F_{0.25,\ 8,\ 106} = 1.31$$

66

substituting into the equation gives

$$\frac{53}{53 + (8 \times 1.31)} = 0.835$$

Or, the 75% lower confidence on mission reliability is

$$0.835 \leq R_m$$

If equation 4 was used to approximate the lower limit, the value obtained would be

$$R_L = \frac{52}{60 + 0.675\sqrt{\frac{60(60-53+1)}{51}}} = 0.838$$

As can be seen, this approximation provides limits that are reasonably close to the exact values.

3.1.3 Bayesian Confidence Limits: If one wants to use a Bayesian approach slightly higher confidence limits will result. The Bayesian $100(1-\alpha)\%$ lower confidence limit on reliability is given by

$$R_L = \frac{(y+1)}{(y+1) + (n-y+1) \; F_{\alpha, 2(n-y+1), 2(v+1)}} \qquad (6)$$

This formula was developed under the assumption that R_m had an equally likely chance of falling anywhere in the interval [0-1], [See 16, for derivation]. Thus, the above limit might be considered a conservative Bayesian limit.

Example 3

Reconsidering example 2 with

n = 60

y = 53

67

A Bayesian limit can be obtained. The required F value is

$$F_{0.25, \ 8, \ 108} = 1.31$$

Then the confidence limit is

$$\frac{54}{54 + (8 \times 1.31)} = 0.837$$

Or,

$$0.837 \leq R_m$$

3.2 Success Testing

In planning a reliability test it is sometimes helpful to know the number of successful missions required to demonstrate a minimum reliability at a desired level of confidence. The formulas for this situation will now be given.

For the special case where y = 0 (i.e. no failures), the lower $100(1-\alpha)\%$ confidence limit on the reliability is

$$R_L = \alpha^{1/n}$$

where

α = the level of significance

n = the sample size (i.e. number of missions)

Then with $100(1-\alpha)\%$ confidence, we can say that

$$R_L \leq R_m$$

where R_m is the true mission reliability.

If we let $C = 1 - \alpha$ be the desired confidence level (i.e. 0.80, 0.90, etc.) then the necessary sample size to demonstrate a desired reliability level is

$$n = \frac{\ln(1-C)}{\ln R} \tag{8}$$

For example, if a mission reliability of $R_m = 0.80$ is to be demonstrated at 90% confidence we have

$$n = \frac{\ln(0.10)}{\ln(0.80)} = 11$$

Thus, 11 missions must be completed with no failures. This is frequently referred to as success testing.

Example 4

A fuel trailer must be tested to demonstrate that it has an 80% mission reliability at 50% confidence. Let's determine the number of missions it must complete without failure in order to demonstrate the required reliability.

Substituting into equation 8 with

$$C = 0.50$$
$$R = 0.80$$

gives

$$n = \frac{\ln 0.50}{\ln 0.80} = 3$$

Or, three successful missions are required.

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